Subalgebra and automorphism structure in universal algebras; a concrete characterization

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§ 1. Introduction

E. T. SCHMIDT [4] has shown that, from an abstract viewpoint, the automorphism structure and subalgebra structure of algebras are completely independent of each other (Theorem 1). The main purpose of this note is to give a concrete version of Schmidt's result where isomorphism is replaced by equality (Theorem 4). We also show how Schmidt's theorem may be derived from the corresponding concrete result.

By an *algebra* $\mathfrak{A}=\langle A, F\rangle$, we mean a *universal algebra* with base set A and a (possibly infinite) set of finitary operations F. We denote by $\operatorname{Aut}\mathfrak{A}$ the automorphism group of $\mathfrak{A}$ and by $\operatorname{Su}\mathfrak{A}$ the closure structure consisting of subsets of A which are subalgebras of $\mathfrak{A}$. For $B\subseteqq A$ we use $\mathbf{C}B$ to denote the subalgebra of $\mathfrak{A}$ generated by B, thus $\mathbf{C}B\in\operatorname{Su}\mathfrak{A}$. $\operatorname{Su}\mathfrak{A}$ is always an algebraic closure structure; when viewed as a partially ordered set $\mathfrak{L}=\langle\operatorname{Su}\mathfrak{A}, \subseteqq\rangle$ is referred to as the *subalgebra lattice* of $\mathfrak{A}$, and this lattice is always compactly generated. Schmidt's result can be stated in the following form:

Theorem 1. [4] *Given any group $\mathfrak{G}$ and any algebraic (compactly generated) lattice $\mathfrak{L}$, there is an algebra $\mathfrak{A}$ with $\mathfrak{G}$ isomorphic to* $\operatorname{Aut}\mathfrak{A}$ *and $\mathfrak{L}$ isomorphic to* $\langle\operatorname{Su}\mathfrak{A}, \subseteqq\rangle$.

§ 2. Concrete characterization

To characterize $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Su}\mathfrak{A}$ *concretely* one must specify for a given set A which subgroups G of the permutation group on A are compatible with various algebraic closure structures L defined on subsets of A, in the sense that $G=\operatorname{Aut}\mathfrak{A}$ and $L=\operatorname{Su}\mathfrak{A}$ for suitable $\mathfrak{A}=\langle A, F\rangle$. Separate concrete characterization theorems for $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Su}\mathfrak{A}$ have been given by other authors. B. JONSSON has characterized the groups of permutations of A which are equal to $\operatorname{Aut}\mathfrak{A}$ for some algebra $\mathfrak{A}=\langle A, F\rangle$, viz.

Theorem 2. [3] *For a subgroup G of the permutation group on a set A to be equal to* $\mathrm{Aut}\,\mathfrak{A}$ *for some algebra with base set A it is necessary and sufficient that G be locally closed in the following sense: if φ is a permutation of A and on each finite subset of A, φ agrees with some $\psi \in G$, then $\varphi \in G$ as well.*

The concrete characterization of $\mathrm{Su}\,\mathfrak{A}$ is due to G. BIRKHOFF and O. FRINK:

Theorem 3. [1] *If L is any algebraic closure structure consisting of subsets of A then $L = \mathrm{Su}\,\mathfrak{A}$ for some algebra $\mathfrak{A} = \langle A, F\rangle$.*

For the combined concrete characterization the most general situation one could expect (complete independence) would be that every locally closed group should be compatible (in the above sense) with every algebraic closure structure definable on A. There is however some interplay between $\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Su}\,\mathfrak{A}$, as the following theorem shows.

Theorem 4. *If G is a group of permutations on A and L is an algebraic closure structure consisting of subsets of A, then there is an algebra $\mathfrak{A} = \langle A, F\rangle$ such that $G = \mathrm{Aut}\,\mathfrak{A}$ and $L = \mathrm{Su}\,\mathfrak{A}$ iff the following conditions are satisfied:*

(1) *G is locally closed;*
(2) *$B \in L$, $\varphi \in G \Rightarrow \varphi(B) \in L$;*
(3) *for each finite subset $X \subseteqq A$, if σ and $\tau \in G$ agree on X, then they agree on CX as well.*

§ 3. Proof of Theorem 4

The necessity of (2) and (3) is obvious, and (1) of course follows from Theorem 2. To prove the converse of the theorem, assume (1), (2) and (3) hold for some A and suitable G and L. We construct an algebra $\mathfrak{A}$ with base set A as follows: There will be two kinds of operations, $f_{(y,a)}$ and F_z, indexed by sets I and J respectively where

$$I = \{(y, a) \mid y \text{ is a finite 1—1 sequence of elements of } A \text{ and } a \in CX - X,\ X = \text{Range } y\}$$

and

$$J = \{z \mid z \text{ is a finite 1—1 sequence of elements of } A\}.$$

The operations are defined in the following way: For $(y, a) \in I$ with the length of $y = n > 0$, $f_{(y,a)}$ is n-ary and

$$f_{(y,a)}(w_0 \dots w_{n-1}) = \begin{cases} \sigma a & \text{if } \quad w = \sigma y \quad \text{for some} \quad \sigma \in G, \\ w_0 & \text{otherwise.} \end{cases}$$

For $(y, a) \in I$ with the length of $y = 0$, $f_{(y,a)}$ is nullary and $f_{(y,a)} = a$. For $z \in J$ of length

n, F_z is n-ary and

$$F_z(w_0 \dots w_{n-1}) = \begin{cases} w_0 & \text{if} \quad w = \sigma y \quad \text{for some} \quad \sigma \in G, \\ w_1 & \text{otherwise.} \end{cases}$$

Note that $f_{(y,a)}$ is well defined since (3) yields $\sigma y = \tau y \Rightarrow \sigma a = \tau a$.

Now let $\mathfrak{A} = \langle A; \langle f_i | i \in I \rangle \cup \langle F_j | j \in J \rangle \rangle$. It will suffice to show that $G = \operatorname{Aut} \mathfrak{A}$ and $L = \operatorname{Su} \mathfrak{A}$.

To see that $L = \operatorname{Su} \mathfrak{A}$, let $B \in \operatorname{Su} \mathfrak{A}$. Note $B \in L$ iff $\mathbf{C}B = B$. Now if $B \notin L$ one has $\mathbf{C}B \neq B$ so there is some $a \in \mathbf{C}B \sim B$. Since L is algebraic there is some finite $X \subseteqq B$ with $\mathbf{C}X = \mathbf{C}B$ and $a \in \mathbf{C}X - X$. If $X = \emptyset$ then $f_{(\emptyset, a)} = a \in B$, so $X \neq \emptyset$. Let y be a $1-1$ sequence whose range is X; then $f_{(y,a)}(y) = a \in B$, contradicting the choice of a. Thus $\operatorname{Su} \mathfrak{A} \subseteqq L$. To establish the opposite inclusion it is only necessary to select $B \in L$ and verify that B is closed under the operations of $\mathfrak{A}$. This is obviously true for operations F_z, $z \in J$. Now let $f_{(y,a)}$ be an operation of $\mathfrak{A}$ of rank n. If $n = 0$, i.e. y = empty sequence, then $a \in \mathbf{C}\emptyset \subseteqq \mathbf{C}B = B$ thus $f_{(y,a)} = a \in B$. If $n > 0$ say $y = \langle y_0 \dots y_{n-1} \rangle$, let X = range y and let $b = \langle b_0, \dots, b_{n-1} \rangle \in B^n$. If $\sigma y \neq b$ for any $\sigma \in G$ then $f_{(y,a)}(b) = b_0 \in B$. If $\sigma y = b$ for some $\sigma \in G$ then $f_{(y,a)}(b) = \sigma a$; but $\sigma(X) \subseteqq B$ so $X \subseteqq \sigma^{-1}(B)$ and $\mathbf{C}X \subseteqq \mathbf{C}\sigma^{-1}B$ thus $a \in \mathbf{C}\sigma^{-1}B$. Further since $B \in L$ we have $\sigma^{-1}B \in L$ so $\mathbf{C}\sigma^{-1}B = \sigma^{-1}B$ and it follows that $\sigma a \in B$ as desired. This shows that $L = \operatorname{Su} \mathfrak{A}$.

It remains only to see that $G = \operatorname{Aut} \mathfrak{A}$. Let $\varphi \in G$. To verify that $\varphi \in \operatorname{Aut} \mathfrak{A}$ one must check that φ is substitutive over each of the operations F_z, and $f_{(y,a)}$. By noting that $y = \sigma z$ iff $\varphi y = \varphi \sigma z$ for $\sigma \in G$, it is clear that $\varphi F_z(y) = F_z(\varphi y)$. The check for $f_{(y,a)}$ is straightforward, and establishes $G \subseteqq \operatorname{Aut} \mathfrak{A}$. Now let φ be a permutation of A, $\varphi \notin G$. We will show $\varphi \notin \operatorname{Aut} \mathfrak{A}$, and it follows that $G = \operatorname{Aut} \mathfrak{A}$. G is locally closed therefore there is a finite set $D \subseteqq A$ such that no member of G agrees with φ on D. Let z be a 1—1 sequence with range D. Then F_z is an $\mathfrak{A}$ operation, and $\varphi F_z(z) = \varphi z_0$, whereas $F_z(\varphi z) = \varphi z_1$. Since z is 1—1 and φ is a permutation, $\varphi F_z(z) \neq F_z(\varphi z)$, thus $\varphi \notin \operatorname{Aut} \mathfrak{A}$. This completes the proof of Theorem 4.

§ 4. Theorem 1 as a corollary

Given a group $\mathfrak{G} = \langle G, \cdot \rangle$ and an algebraic lattice $\mathfrak{L}$, we exhibit a set A, a group of permutations of A, $\mathfrak{G}^+ \cong \mathfrak{G}$, and an algebraic closure structure, L^+, on subsets of A, with $\langle L^+, \subseteqq \rangle \cong \mathfrak{L}$ (lattice isomorphism). Further $\mathfrak{G}^+$ and L^+ will satisfy (1)(2)(3) of Theorem 4 so that one may conclude $\mathfrak{G} \cong \operatorname{Aut} \mathfrak{A}$ and $\mathfrak{L} \cong \langle \operatorname{Su} \mathfrak{A}, \subseteqq \rangle$ for some algebra $\mathfrak{A}$. The abstract theorem in a way asserts the existence of a concrete object; the only problem is first to find a suitable set for the application of Theorem 4. As a first step note that by well-known lattice theoretic considerations, we may assume that (within isomorphism) $\mathfrak{L}$ is a lattice of subsets of some set B, and furthermore that the zero element of $\mathfrak{L}$ is the null set. Now let $A=\{(x, y)|x\in B, y\in G\}$. For each $g\in G$ define $g^+:A\to A$ by $g^+(x,y)=(x, g\cdot y)$. Let $G^+=\{g^+|g\in G\}$, and $G^+=\langle G^+, 0\rangle$; clearly $\varphi:\mathfrak{G}\to\mathfrak{G}^+$ by $\varphi(g)=g^+$ is a group isomorphism. Note that $\mathfrak{G}\cong\mathrm{Aut}\,\langle G; f_g\rangle_{g\in G}$ where f_g is the unary operation "right multiplication by g". Since $\mathrm{Aut}\,\langle G; f_g\rangle_{g\in G}$ is locally closed (by Theorem 2) it follows that $\mathfrak{G}^+$ is locally closed in A^A. Now for each $P\in L$ let $P^+=\{(x,y)\in A|x\in P, y\in G\}$, and set $L^+=\{P^+|P\in L\}$. L^+ is an algebraic closure structure composed of subsets of A, and one easily verifies $\langle L^+, \subseteqq\rangle\cong\mathfrak{L}$ (lattice isomorphism). It is clear from the method of construction that $\mathfrak{G}^+$ and L^+ satisfy (1) and (2) of theorem 4. To see that (3) holds as well, let $X\subseteqq A$. If $X=\emptyset$ then (3) holds vacuously since $\emptyset\in L^+$. If X is finite and non-empty, say $(x_0, y_0)\in X$, suppose $g^+, h^+\in G^+$ agree on X. Then $y_0\cdot g=y_0\cdot h$, but $y_0\in G$ so $g=h$ and thus $g^+=h^+$ so of course (3) is satisfied.

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